# Prescribing Gaussian Curvature on S$^2$(Solutions for Nonlinear Elliptic Equations)



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# Prescribing Gaussian Curvature on $S^2$

by

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On the two sphere $S^2 = \{ x=(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2+x_2^2+x_3^2=1 \}$ with standard metric $ds_0^2 = dx_1^2+dx_2^2+dx_3^2$, when the metric is subjected to the conformal change $ds^2 = e^{2u}ds_0^2$, the Gaussian curvature $K(x)$ of the new metric $ds^2$ is

$$K(x) = (1-\Delta u)e^{-2u}, \tag{1}$$

where $\Delta$ denotes the Laplacian relative to the standard metric. The inverse problem raised by Nirenberg is : which function $K$ on $S^2$ can be prescribed so that (1) has a solution $u$ on $S^2$?

We can rewrite (1) as

$$\Delta u + K(x)e^{2u} = 1 \qquad \text{on } S^2. \tag{2}$$

Let $d\mu$ denotes the standard surface measure on $S^2$. Integrating (2) over the whole sphere, we obtain an obvious necessary condition

$$\int_{S^2} Ke^{2u}d\mu = 4\pi. \tag{3}$$

Hence $K$ must be positive somewhere. Kazdan and Warner [5] found some other necessary condition by integration by parts. For each eigenfunction $x_j$ with $\Delta x_j+2x_j = 0$ $(j=1,2,3)$, the Kazdan-Warner condition state that

$$\int_{S^2} \langle \nabla K, \nabla x_j \rangle e^{2u}d\mu = 0 , \quad j = 1,2,3. \tag{4}$$

Moser [6] was the first to prove that if $K(\cdot)$ is an even function on $S^2$, that is, $K(x) = K(-x)$ for all $x \in S^2$, then

(2) has a solution. Recently, many sufficient conditions were discovered. We refer the reader to Hong [4], Chen and Ding [3], Chang and Yang [1,2].

In this paper, we consider the case that K is rotationally symmetric, that is, K is a function of $x_3$ only. Hong [4] considered this case and established some existence theorems. The method used by Hong is the variational method. In case K is a function of $x_3$, if we are looking for solutions u depending only on $x_3$, then (2) becomes an ordinary differential equations. It is the purpose of this paper to treat (2) by using the standard techniques of ordinary differential equations.

We assume that K is a function of $x_3$ only and we are looking for solutions u depending only on $x_3$. Let $x_3 = z$. Then (2) becomes

$$\frac{d}{dz}[(1-z^2)\frac{du}{dz}] + K(z)e^{2u} = 1, \qquad z \in [-1,1]. \tag{5}$$

Let $z = \frac{r^2-1}{r^2+1}$ and $K_1(r) = K(\frac{r^2-1}{r^2+1})$. We consider the following initial value problem

$$\text{(b)} \qquad \begin{cases} v''(r) + \frac{1}{r}v'(r) + K_1(r)e^{2u(r)} = 0, & r \in (0,\infty) \\ v(0) = \alpha, \quad v'(0) = 0. \end{cases}$$

Let I be the set of $\alpha$ such that (6) has a unique solution $v(r;\alpha)$ on $[0,\infty)$. Our first main result is

<u>Theorem A</u>. Assume that K is Hölder continuous on $S^2$. Then (5) has a regular solution u(z) on [-1,1] if and only if there exists an $\alpha \in I$ such that

$$\int_0^\infty K_1'(r)r^2e^{2v(r;\alpha)}dr = 0 \tag{7}$$

and

$$\int_0^\infty rK(r)e^{2v(r;\alpha)}dr > 0. \tag{8}$$

In general, we do not have much informations about the solutions $v(r;\alpha)$. Thus it is not an easy manner to verify (7).

Let $K_2(r) = K(\frac{1-r^2}{1+r^2})$. Our second main result is

<u>Theorem B</u>. Assume that $K_1$ and $K_2$ are smooth functions on [0,1] and change sign finite times. Then (5) has a regular solution u(z) on [-1,1] if any one of the following statements holds:

(i) $K_1(r) = K_2(r)$ for $0 \le r \le 1$ and $K_2$ is positive somewhere (Moser [6]).

(ii) $K_1(0) > 0$, $K_2(0) > 0$ and $K_1'(0)\cdot K_2'(0) > 0$.

(iii) $K_1(0) > 0$, $K_2(0) > 0$, $K_1'(0) = 0$ and $K_1''(0)\cdot K_2'(0) > 0$.

(iv) $K_1(0) > 0$, $K_2(0) > 0$, $K_1'(0) = K_2'(0) = 0$ and $K_1''(0) \cdot K_2''(0) > 0$.

(v) $K_1(0) > 0$, $K_1'(0) > 0$ and $K_2(0) \le 0$.

(vi) $K_1(0) > 0$, $K_1'(0) = 0$, $K_1''(0) > 0$ and $K_2(0) \leq 0$.

(vii) The roles of $K_1$ and $K_2$ change in (iii), or (v), or (vi).

(viii) $\max\{K_1(0), K_2(0)\} \leq 0$ and $K$ is positive somewhere [Hong, 4].

We shall sketch the proofs of Theorem A and part of Theorem B and leave details to Cheng and Smoller [7].

Sketch of proof of Theorem A. First we need some lemmas.

<u>Lemma 1</u>. Assume that $K_1(r) > 0$ for $r \geq r_0$ and $K_1(r) \sim r^{\ell}$ at $\infty$ for some real number $\ell$. Then

$$\frac{\ell+2}{2} < \int_0^\infty sK_1(s)e^{2v(s;\alpha)}ds < \infty$$

for all $\alpha \in I$. (We use the notation "$f \sim g$ at $\infty$" to denote that "there exist two positive constants $C_1$, $C_2$ such that $C_1 g \geq f \geq C_2 g$ at $\infty$".)

<u>Lemma 2</u> Suppose that $K_1(r) \leq 0$ for $r \geq r_0$ and $K_1(r) \sim -r^{\ell}$ at $\infty$ for some real number $\ell$. Then

$$\int_0^\infty sK_1(s)e^{2v(s;\alpha)}ds \geq \frac{\ell+2}{2}$$

for every $\alpha \in I$. In particular if $K(r) \leq 0$ for all $r > 0$, then

$$0 > \int_0^\infty sK(s)e^{2v(s;\alpha)}ds \geq \frac{\ell+2}{2}.$$

Let

(9) $$A_\alpha(r) = (1+rv'(r;\alpha))^2 + K_1(r)r^2e^{2v(r;\alpha)}.$$

Lemma 3. Suppose that $K_1(r) \sim r^{-\ell}$ or $K_1(r) \sim -r^{\ell}$ at $\infty$ for some real number $\ell$. Then for each $\alpha \in I$,

(i) $A'_\alpha(r) = K'_1(r)r^2e^{2v(r;\alpha)}$,

(ii) $$\lim_{r\to\infty} A_\alpha(r) = (1-\int_0^\infty sK_1(s)e^{2v(s;\alpha)}ds)^2$$
$$= 1 + \int_0^\infty K'(s)s^2e^{2v(s;\alpha)}ds.$$

Now we can sketch the proof of Theorem A. Suppose (5) has a regular solution $u(z)$ on $[-1,1]$. Let $z = \frac{r^2-1}{r^2+1}$ and $v(r) = u(z) - \log(\frac{1+r^2}{2})$. Then $v(r)$ satisfies (6) with $v(0) = u(-1)$ and $v(r) = -2\log r + O(1)$ at $\infty$. It is easy to see that

$$2 = \int_0^\infty sK_1(s)e^{2v(s;\alpha)}ds, \qquad \alpha = u(-1).$$

Hence

$$(1 - \int_0^\infty sK_1(s)e^{2v(s;\alpha)}ds)^2 = 1$$
$$= 1 + \int_0^\infty K'_1(r)r^2e^{2v(r;\alpha)}dr.$$

This proves that $\alpha = u(-1) \in I$ and (7) and (8) hold.

Conversely, if there exists $\alpha \in I$, such that, (7) and (8) holds. Then from Lemmas 1,2 and 3, we have

$$\int_0^\infty sK_1(s)e^{2v(s;\alpha)}ds = 2$$

and

$$v(r;\alpha) = -2\log r + C \qquad \text{for } r \text{ large.}$$

Let $z = \frac{r^2-1}{r^2+1}$ and $u(z) = v(r;\alpha) + \log(\frac{1+r^2}{2})$. Then $u(z)$ is a regular solution of (5). The proof is complete.

We only sketch the proof of Theorem B under the following assumptions. We assume that $K_1(r) = K_2(r) \geq 0$ for all $r \in [0,1]$ and $K_1(0) > 0$. That is, we prove the following theorem.

Theorem B'. Assume that $K_1(r) = K_2(r) \geq 0$ for all $r \in [0,1]$ and $K_1(0) > 0$. Then (5) has a regular solution $u(z)$ on $[-1,1]$.

We need one lemma. Consider first

$$(10) \qquad \begin{cases} v'' + \frac{1}{r}v' + K(r)e^{2v(r)} = 0 & r \in [0,1] \\ v(0) = \alpha,\ v'(0) = 0. \end{cases}$$

Let $J$ be the $\alpha$'s such that (10) has a solution $v(r;\alpha)$ on $[0,1]$.

Lemma 4. Assume that $K(r) \geq 0$ for all $r \in [0,1]$ and $K(0) > 0$. Then $J = (-\infty,\infty)$,

$$(11) \qquad \begin{aligned} v(1;\alpha) &= \alpha - O(e^{2\alpha}) \\ v'(1;\alpha) &= O(e^{2\alpha}) \end{aligned}$$

for $\alpha \to -\infty$, and

(12)
$$v(1;\alpha) = -\alpha + O(1)$$
$$v'(1;\alpha) = -2 + o(1)$$

for $\alpha \to \infty$.

The proof of this lemma is quite long. We refer the interested reader to Cheng and Smoller [7].

Now we can sketch the proof of Theorem B'. First we consider

(13)
$$\begin{cases} \frac{d}{dz}[(1-z^2)\frac{d\tilde{v}}{dz}] + K(z)e^{2\tilde{v}} = 1, & z \in [-1,0] \\ \tilde{v}(-1) = \tilde{\alpha}. \end{cases}$$

We let $\tilde{I}_1$ to denote the set of numbers $\tilde{\alpha}$, such tht, (13) has a solution $\tilde{v}\ (z;\tilde{\alpha})$ on $[-1,0]$. Let

$$A_1 = \{(\tilde{v}(0,\tilde{\alpha}),\tilde{v}'(0,\tilde{\alpha})) \in \mathbb{R}^2 : \tilde{\alpha} \in \tilde{I}_1\}.$$

Similarly, let

(14)
$$\begin{cases} \frac{d}{dz}[(1-z^2)\frac{d\tilde{w}}{dz}] + K(z)e^{2\tilde{w}} = 1, & z \in [0,1] \\ \tilde{w}(-1) = \tilde{\beta}. \end{cases}$$

Let $\tilde{I}_2$ be the set of numbers $\tilde{\beta}$ such that (14) has a solution $\tilde{w}(z;\tilde{\beta})$ on $[0,1]$. Let

$$A_2 = \{(\tilde{w}(0;\tilde{\beta}),\ \tilde{w}'(0;\tilde{\beta})) \in \mathbb{R}^2 : \tilde{\beta} \in \tilde{I}_2\}.$$

It is easy to see that (5) has a regular solution $u(z)$ if and only if $A_1 \cap A_2 \neq \phi$.

Now let $z = (r^2-1)/(r^2+1)$ and $\tilde{v}(z) = v(r) + \log(\frac{1+r^2}{2})$. Then $v(r)$ satisfies

$$(15) \quad \begin{cases} v''(r) + \frac{1}{r}v'(r) + K_1(r)e^{2v(r)} = 0, & r \in [0,1] \\ v(0) = \tilde{\alpha} + \log 2 \equiv \alpha,\ v'(0) = 0,\ \alpha \in I_1 = \tilde{I}_1 + \log 2, \end{cases}$$

where $K_1(r) = K(\frac{r^2-1}{r^2+1})$. Hence we have

$$A_1 = \{(v(1;\alpha),\ (1+v'(1;\alpha)) : \alpha \in I_1\}.$$

Similarly, let $z = \frac{1-r^2}{1+r^2}$ and $\tilde{w}(z) = w(r) + \log(\frac{1+r^2}{2})$. Then $w(r)$ satisfies

$$(16) \quad \begin{cases} w''(r) + \frac{1}{r}w'(r) + K_2(r)e^{2w(r)} = 0, & r \in [0,1] \\ w(0) = \tilde{\beta} + \log 2 \equiv \beta,\ w'(0) = 0,\ \beta \,\delta\, I_2 = \tilde{I}_2 + \log 2. \end{cases}$$

Then we have

$$A_2 = \{(w(1;\beta),\ -(1+w'(1;\beta)) : \beta \in I_2\}.$$

Now from Lemma 4, $I_1 = I_2 = (-\infty,\infty)$, then curve corresponding to the part of $A_1$ when $\alpha \to -\infty$ approaches the curve $y = 1$, $x \to -\infty$ and the curve corresponding to the part of $A_1$ when $\alpha \to \infty$ approaches the curve $y = -1$, $x \to -\infty$. Hence the curve $A_1$ must intersect the line $y = 0$ at some point. From the assumption, $K_1 = K_2$, we conclude that $A_2$ is the mirror images of $A_1$ with respect to the mirror $y = 0$. Hence $A_1 \cap A_2 \neq \phi$. This completes the sketch of proof of Theorem B'.

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